



# A stepwise approach to formalizing mathematics

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(work in progress)

Types for Mathematics / Libraries of Formal Mathematics

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# Motivation

Study the process of formalizing of informal math texts

- (Input) Informal math text in CML (e.g. English)
- (Output) Formalization in a proof assistant

We want:

- Reliability
- Usability by Humans
- Computer Assistance

# Overview

- Stepwise Formalization with Weak Type Theory
- Translating Weak Type Theory into Type Theory with open terms.
- Examples and Problems



# Stepwise Formalization using Weak Type Theory

# What is Weak Type Theory

Formal language in the Automath tradition

- Identify structure: definitions, statements, etc.
- Rich in math primitives

$$\lambda x:A.M; \sum_{x:A} f(x); \iota x:A.\phi(x); \{x:A|\phi(x)\} \dots$$

- No built-in logic
- Enforces linguistic correctness via weak types:

adjective, noun, term, set, statement

# Examples

adjective

$Adj_{x:N}.(0 < x) \rightarrow$  "positive"

$Adj_{x:N}.(∃n:N.2n = x) \rightarrow$  "even"

statement

$∀k:Positive\ N.((2^k - 1) \text{ is prime}) \Rightarrow (2^{k-1}(2^k - 1) \text{ is perfect})$

term  $\sqrt{b^2 - 4ac}$ ,  $\lambda x:N.x + x$

noun

$Noun_{x:N} \text{ prime}(n) \rightarrow$  "a prime number"

$\text{positive even } N \rightarrow$  "a positive even number"

# Weak Type Theory

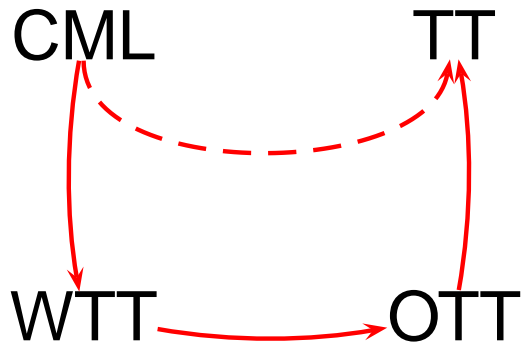
## The Language

term	$\mathcal{T}$	$::=$	$x \mid c(\vec{\mathcal{P}}) \mid \lambda_{\mathcal{Z}}\mathcal{T} \mid \dots$
adjective	$\mathcal{A}$	$::=$	$c(\vec{\mathcal{P}}) \mid \text{Adj}_{\mathcal{Z}}(\mathcal{S}_p) \mid \dots$
noun	$\mathcal{N}$	$::=$	$c(\vec{\mathcal{P}}) \mid \text{Noun}_{\mathcal{Z}}(\mathcal{S}_p) \mid \text{Abst}_{\mathcal{Z}}(\mathcal{T}/\mathcal{N}/\mathcal{S}_s) \mid \mathcal{AN} \mid \mathcal{S}_s \downarrow \mid \dots$
set	$\mathcal{S}_s$	$::=$	$x \mid c(\vec{\mathcal{P}}) \mid \text{Set}_{\mathcal{Z}}(\mathcal{S}_p) \mid \mathcal{N} \uparrow \mid \dots$
statement	$\mathcal{S}_p$	$::=$	$x \mid c(\vec{\mathcal{P}}) \mid \mathcal{S}_p \rightarrow \mathcal{S}_p \mid \forall_{\mathcal{Z}}(\mathcal{S}_p) \mid \exists_{\mathcal{Z}}(\mathcal{S}_p) \mid \mathcal{T} \text{ is } \mathcal{A} \mid \mathcal{T} \text{ is } \mathcal{N} \mid \dots$
binder	$\mathcal{B}$	$::=$	$B_{\mathcal{Z}}.(\mathcal{T}/\mathcal{A}/\mathcal{N}/\mathcal{S}_s/\mathcal{S}_p)$
argument	$\mathcal{P}$	$::=$	$\mathcal{S}_s \mid \mathcal{S}_p \mid \mathcal{T}$
declaration	$\mathcal{Z}$	$::=$	$x:\text{SET} \mid x:\text{STAT} \mid x:\mathcal{N} \mid x:\mathcal{S}_s$
context	$\Gamma$	$::=$	$\emptyset \mid \Gamma, \mathcal{Z} \mid \Gamma, \mathcal{S}_p$
definition	$\mathcal{D}$	$::=$	$c(\vec{x}) := (\mathcal{T}/\mathcal{S}_p/\mathcal{S}_s/\mathcal{A}/\mathcal{N})$
line	$l$	$::=$	$\Gamma \triangleright \mathcal{D} \mid \Gamma \triangleright \mathcal{S}_p$
book	$\mathcal{B}$	$::=$	$\emptyset \mid \mathcal{B} \circ l$

# The Formalization Path

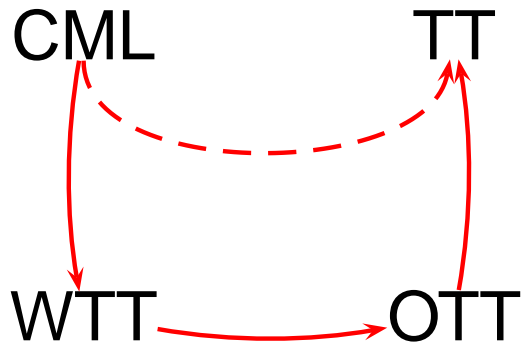


# The Formalization Path



TT: Type Theory  
OTT: TT with Open terms

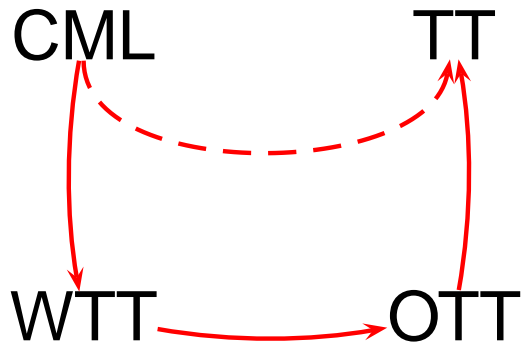
# The Formalization Path



TT: Type Theory  
OTT: TT with Open terms

- Write down the informal text into the formal language, considering only very weak (linguistic) correctness criteria

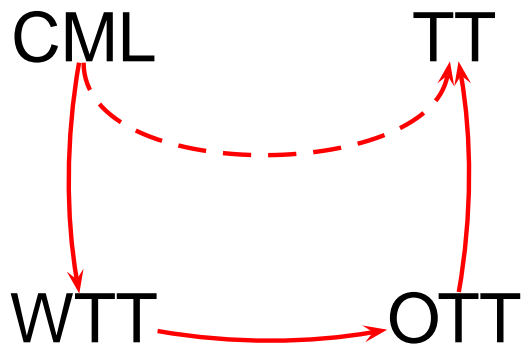
# The Formalization Path



TT: Type Theory  
OTT: TT with Open terms

- Write down the informal text into the formal language, considering only very weak (linguistic) correctness criteria
- Give "semantics" of the formal text. Here we make most of the "formalization" choices.

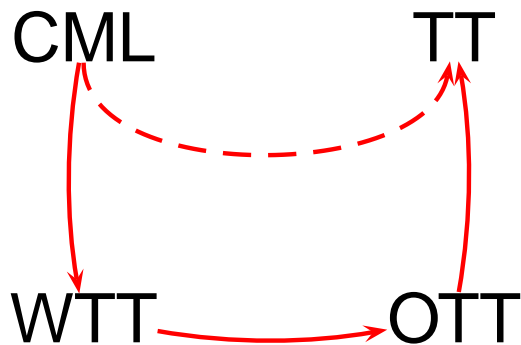
# The Formalization Path



TT: Type Theory  
OTT: TT with Open terms

- Write down the informal text into the formal language, considering only very weak (linguistic) correctness criteria
- Give "semantics" of the formal text. Here we make most of the "formalization" choices.
- Fill in implicit/missing details

# Benefits



Reliability

Added value to the user

- Remain close to the original
- Getting faster "to the point"
- Opportunities for automation/checks
- Complements existing technologies
- Library of "models" of WTT



# **Translating WTT into Type Theory with Open Terms**

# Translating Books and Lines

Books, lines:

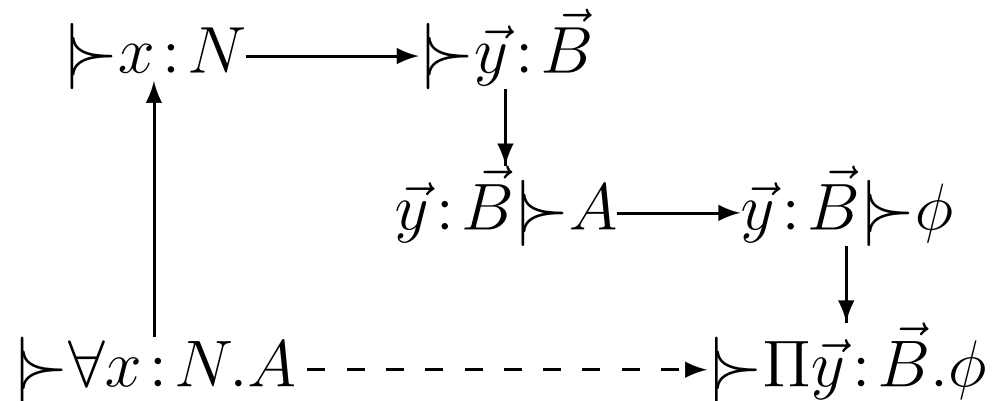
$$\mathcal{B} := \emptyset \mid \mathcal{B}; \Delta \triangleright c(\vec{x}) := M \mid \mathcal{B}; \Delta \triangleright \psi$$

Books are mapped into TT contexts containing constants, meta-variables and definitions

$$\begin{array}{ccc} \Sigma \vdash \Delta & \longrightarrow & \Sigma_1 \vdash \Theta \\ \Sigma_1, \Theta \vdash M & \longrightarrow & \Sigma_2, \Theta \vdash t \end{array}$$

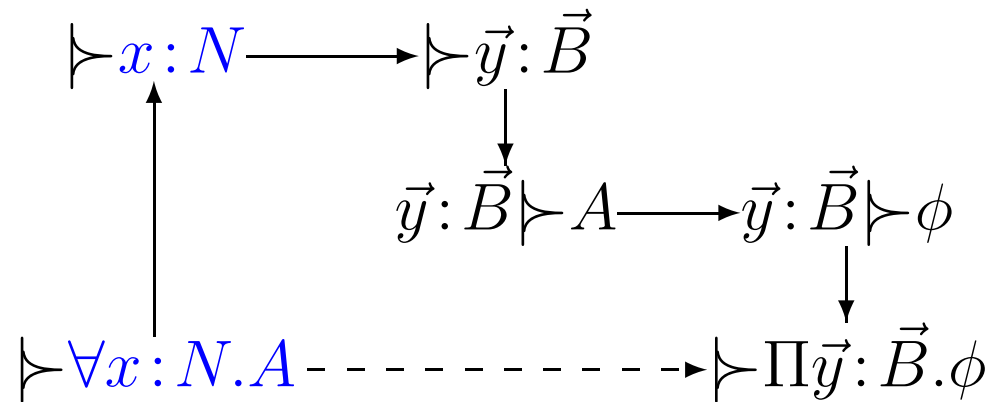
$\Delta \triangleright c(\vec{x}) := M$  is translated into  $\Sigma, c(\Theta) := t : A$

# Rule Diagrams



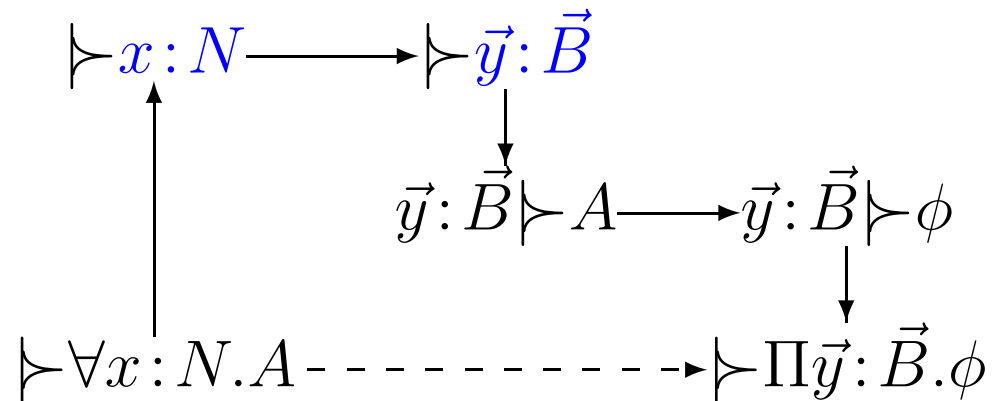
$$\begin{array}{c}
 \Gamma_1 \vdash x : N \longrightarrow \Gamma_2 \vdash \vec{y} : \vec{B} \\
 \Gamma_2, \vec{y} : \vec{B} \vdash A \longrightarrow \Gamma_3, \vec{y} : \vec{B} \vdash \phi \\
 \hline
 \Gamma_1 \vdash \forall x : N. A \longrightarrow \Gamma_3 \vdash \Pi \vec{y} : \vec{B}. \phi
 \end{array}$$

# Rule Diagrams



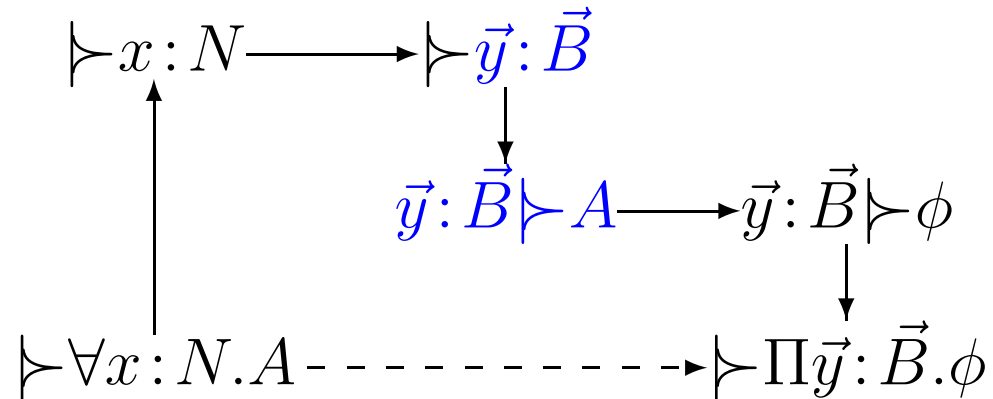
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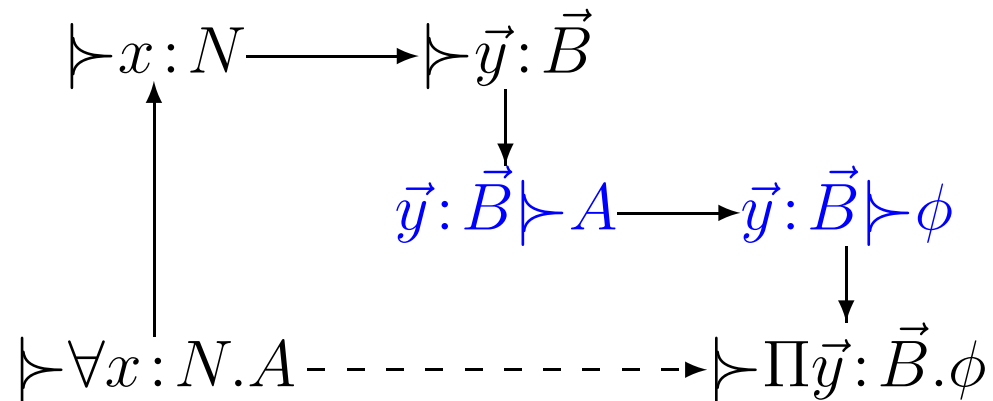
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# Rule Diagrams



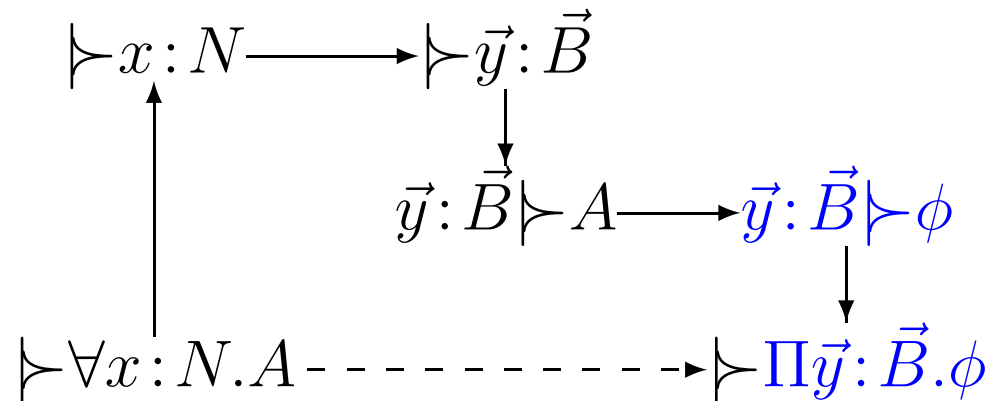
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 \hline
 \Gamma_1 \vdash \forall x : N . A \longrightarrow \Gamma_3 \vdash \Pi \vec{y} : \vec{B} . \phi
 \end{array}$$

# Context Preservation

$$\begin{array}{c} \Gamma, \vec{s} : \vec{P} \vdash \vec{t} : \vec{Q} \\ \uparrow \quad \downarrow \\ \Gamma, \vec{x} : \vec{A} \vdash \vec{y} : \vec{B} \end{array}$$

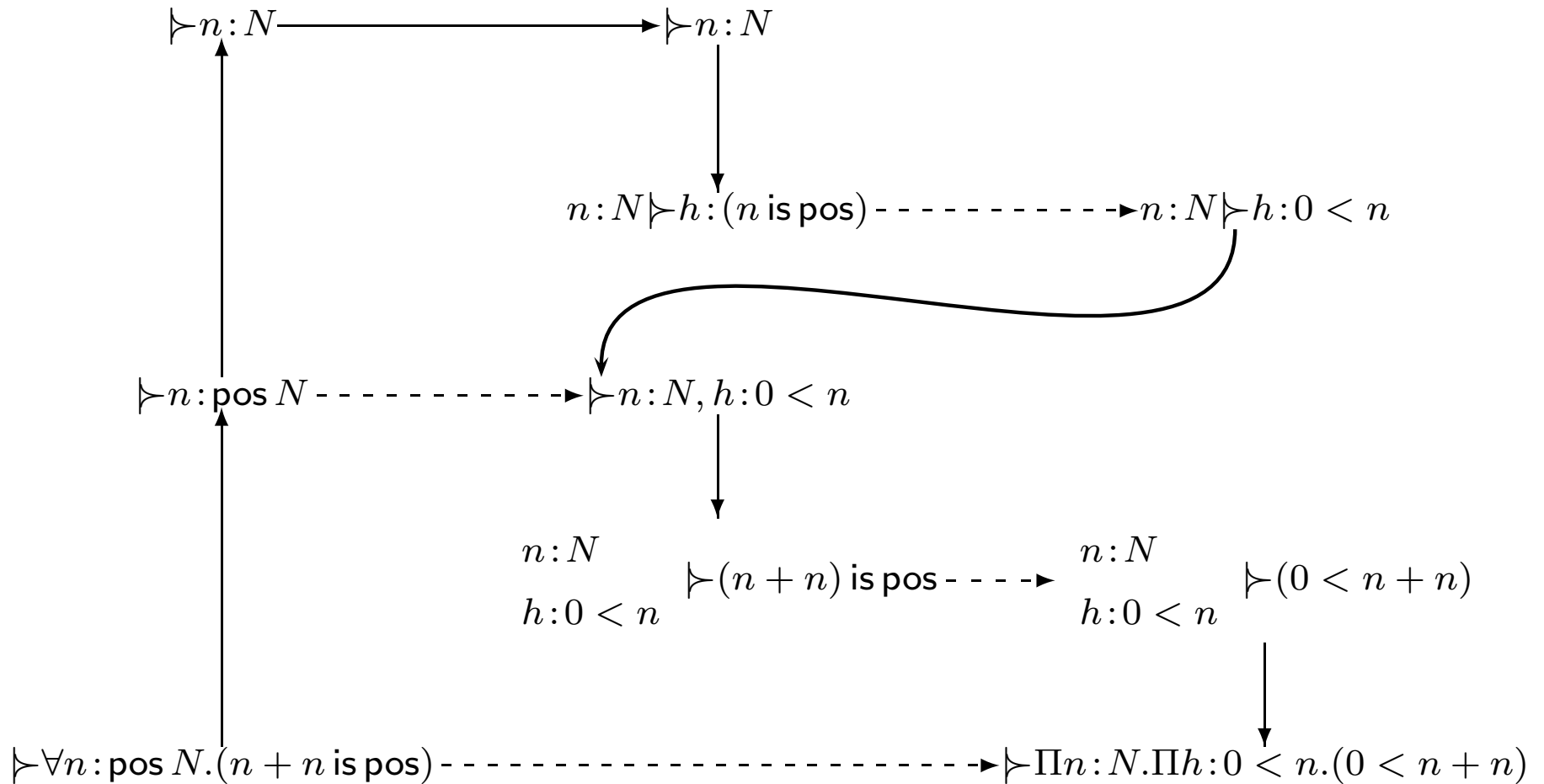
Upward Arrows preserve the implicit context. Downward arrows may add meta-variables to the context

$$\Gamma_1, \vec{s} : \vec{P} \vdash \vec{t} : \vec{Q} \dashrightarrow \Gamma_2, \vec{x} : \vec{A} \vdash \vec{y} : \vec{B}$$

Horizontal Arrows preserve the object-level part of the contexts and may weaken the meta-level by adding new meta-variables

# Example

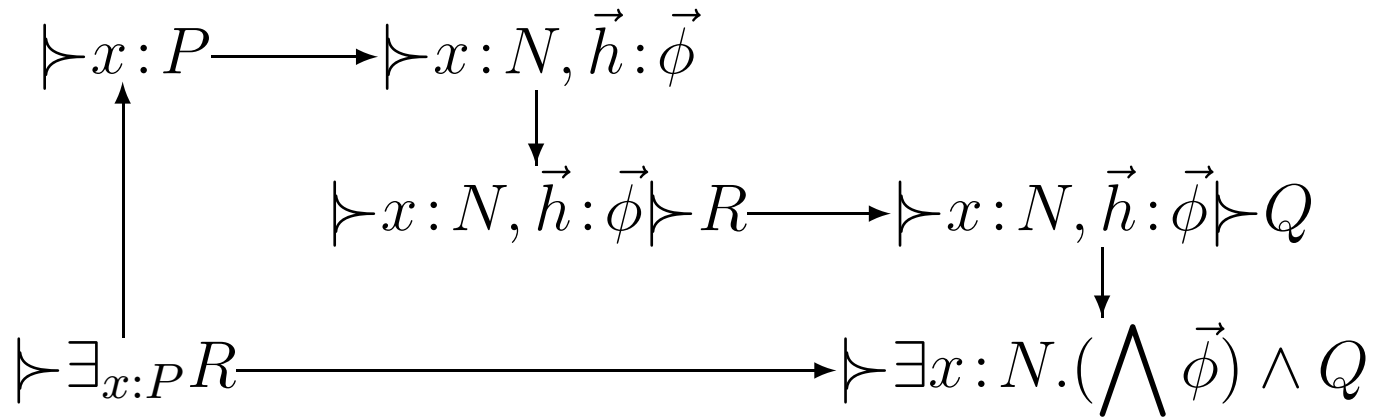
$\text{pos} := \text{Adj } n:N.0 < n$



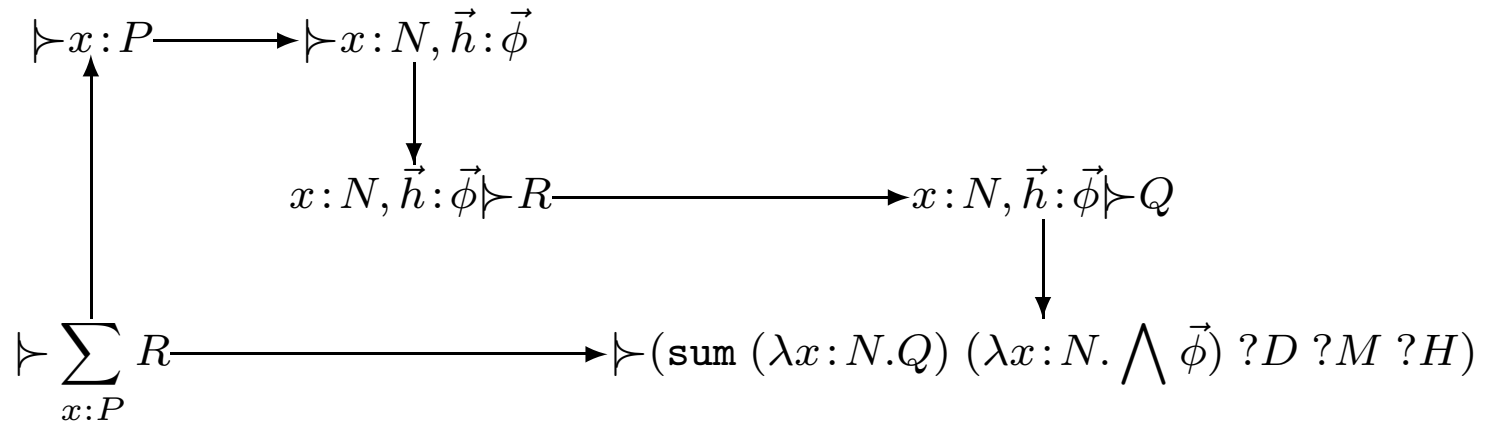
# Other rules

$$\begin{array}{ccc}
 \vdash x : N & \longrightarrow & \vdash \vec{x} : \vec{A} \\
 \uparrow & & \downarrow \\
 \vdash f : M \rightarrow N & \longrightarrow & \vdash f_i : \Pi \vec{x} : \vec{A}. \vec{B}[(f_j \vec{x}) / y_j]_{j < i} \\
 & & \downarrow \\
 & & \vec{x} : \vec{A} \vdash y : N \longrightarrow \vec{x} : \vec{A} \vdash \vec{y} : \vec{B}
 \end{array}$$

# Other rules



# Other rules



$\Pi f:(N \rightarrow N)$   
 $\Pi \phi:(N \rightarrow \text{STAT})$   
 $\text{sum} : \Pi d:(\forall n:N. \phi(n) + \neg \phi(n))$   
 $\Pi M:N$   
 $\Pi h:(\forall n:N. \phi(i) \rightarrow i \leq M)$   
 $N$

# Problems and Future Work

- How to define correctness criteria in general?
- How strong unification/matching do we need?
- Other models
- Implementation and Larger case study