Small TYPES workshop
Nijmegen, 1-2 Nov 04

DEVELOPING MATHEMATICS OVER TYPE THEORY: TRIPARTITION OF TASKS

Giovanni Sambin
(Padova)

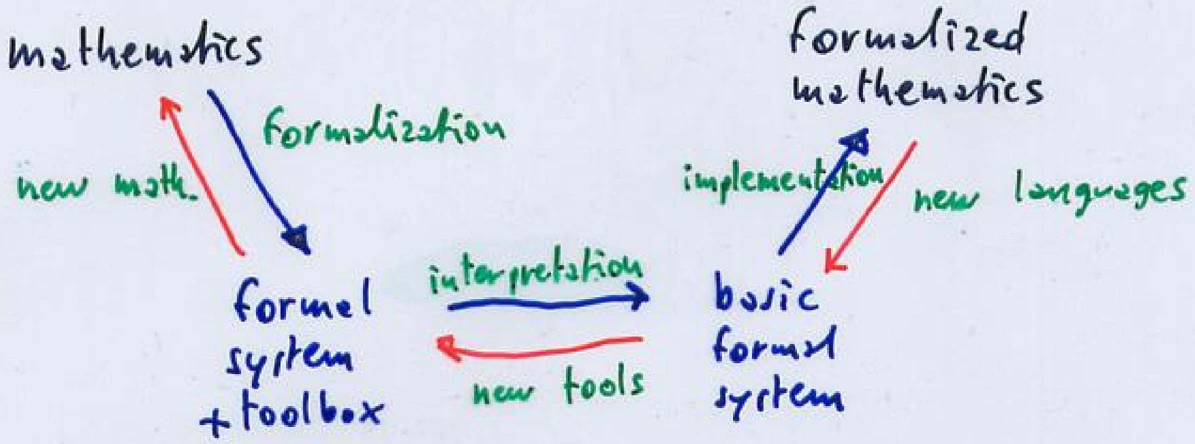
- partly with Milly Maietti

- usual view

mathematics $\xrightarrow[\text{implementation}]{}$ formalized math.

my view (and experience)

math. $\longrightarrow$ formalizable math. $\dashrightarrow$ formalized math.



prove theorems
give definitions
new mathematics
choose foundation
objects

write programs
new languages
new implementations
expressions

build tools to
forget safely
new foundations
keep objects
and expressions
in context

partition of tasks means:
- accumulation of knowledge
- dynamics
- modularity (... a Citroen body with Alfa Romeo engine)
- re-usability

Proofs-as-programs

the machine proves $\exists z\,(E(z)=0)$

the user waits a button: "give me one solution"

my choice was for Martin-Löf type theory

Toolbox

forget some information safely
preserve computational content

subsets $\quad S$ set

$U \subseteq S \equiv U(x)$ prop $(x \in S)$

$a \,\varepsilon_S\, U \equiv Id(S,a,a)\,\&\,U(a)$

$a \,\varepsilon\, U$ true iff $a \in S$ and $U(a)$ true

<u>all</u> (intuitionistic) properties of subsets derive from:

$(\forall a \in S)(a \,\varepsilon_S\, U \leftrightarrow U(a)) \qquad \dfrac{a \,\varepsilon_S\, U}{a \in S} \quad \dfrac{a \,\varepsilon_S\, U}{U(a) \text{ true}}$

other tools:

quotients
finite subsets (of all kinds)
extensional equality

new foundation $\longrightarrow$ new mathematics

formal cover $\triangleleft$ $\rightarrow$ inductive methods in topology

new notation $U \between Z \equiv \exists a (a \varepsilon U \,\&\, a \varepsilon Z)$

dual to $U \subseteq Z \equiv \forall a (a \varepsilon U \rightarrow a \varepsilon Z)$

brings to

duality between open and closed

$$x \varepsilon \mathrm{int}\, D \equiv \exists a (x \Vdash a \,\&\, \mathrm{ext}(a) \subseteq D) \equiv x \varepsilon \mathrm{ext}\, \Box D$$

$$x \varepsilon \mathrm{cl}\, D \equiv \forall a (x \Vdash a \rightarrow \mathrm{ext}(a) \between D) \equiv x \varepsilon \mathrm{rest} \Diamond D$$

symmetry between pointwise and pointfree

$$\mathrm{int} \rightsquigarrow \mathcal{J} \quad \ltimes$$

$$\mathrm{cl} \rightsquigarrow \mathcal{A} \quad \triangleleft$$

positivity relation $\ltimes$ dual to $\triangleleft$

co-inductive methods in topology

overlap algebras put topology in algebraic terms

new math. $\longrightarrow$ new applications

see Hancock-Hyvernat, Programming as applied basic topology

$$
\begin{tikzcd}
 & \begin{array}{c}\text{classical logic}\\ \text{ZFC}\end{array} & \\
\begin{array}{c}\text{Martin-Löf}\\ \text{type theory}\\ \text{predicative}\end{array} \arrow[ur, "\mathcal{P}X"] & & \begin{array}{c}\text{topos theory}\\ \text{impredicative}\end{array} \arrow[ul, "\text{AC}"'] \\
 & \begin{array}{c}\text{mTT}\\ \text{minimal type theory}\end{array} \arrow[ul, "\begin{array}{c}\text{AC}\\ \text{prop-as-set}\end{array}"'] \arrow[ur, "\mathcal{P}X\ \text{set}"'] &
\end{tikzcd}
$$

- no formal-system-bricolage, we want to understand
- independence of logic from set theory
- everybody free to keep his Weltanschauung

  computational view: mTT preserves computational content

  geometric view: $\mathcal{P}X$ set, points form a set proofs-as-programs

Heyting semantics of intuitionistic logic

proof = method to give a verification

axiom of choice ^{AC} is valid

$$(\forall x \in A)(\exists y \in B)\, R(x,y) \rightarrow (\exists f \in A \rightarrow B)\, \forall x \in A\ R(x, fx)$$

automatic extraction of programs

proposition-as-set interpretation

Church thesis CT is valid

but:

AC + CT valid + extensionality + $\mathcal{P}X$ set $\vdash 0 = 1$

$f = g$ iff $\forall x\,(fx = gx)$

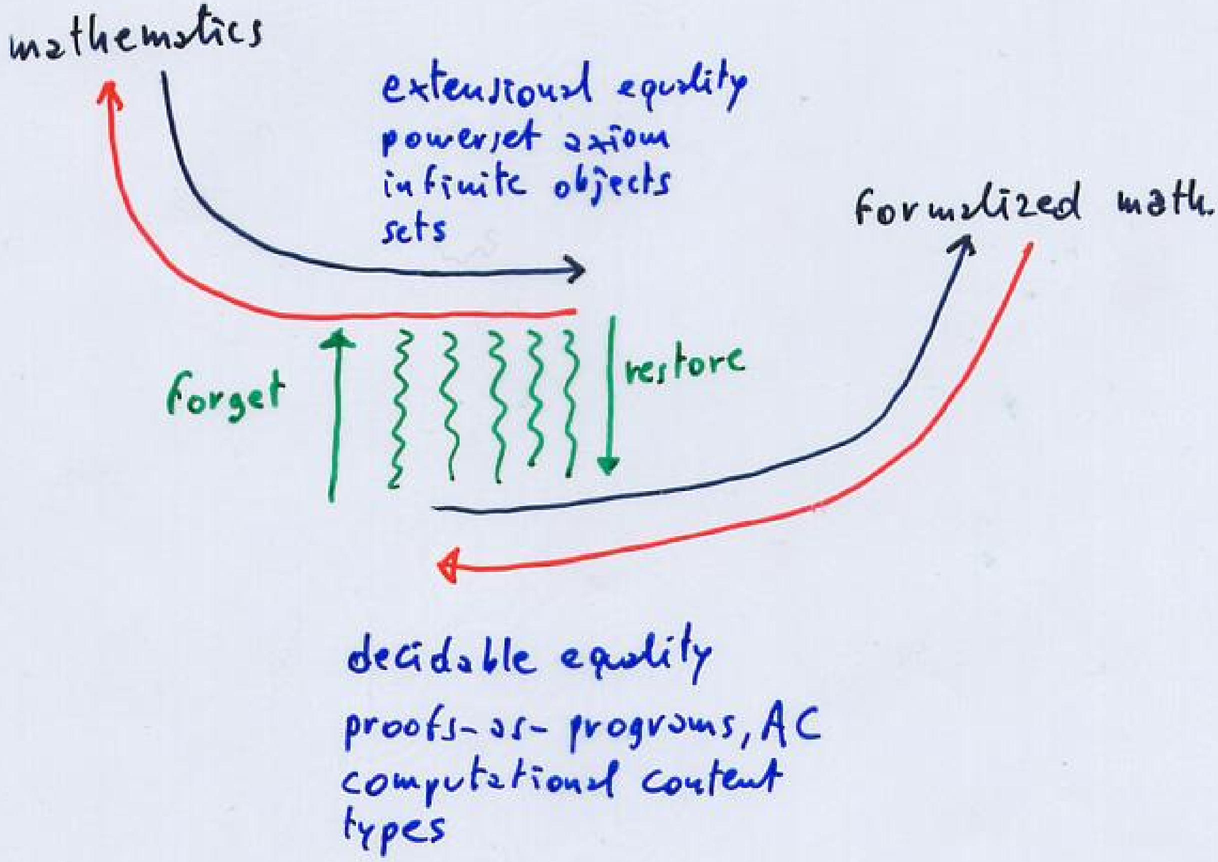
more precisely:

AC + extensionality $\vdash$ undecidability

no type checking

Heyting semantics valid + Church Th. $\vdash$ formalized CT

AC + $\mathcal{P}X$ set $\vdash P \vee \neg P$


mathematics
extensional equality
powerset axiom
infinite objects
sets
formalized math.
forget
restore
decidable equality
proofs-as-programs, AC
computational content
types

proposal for mTT

1. logic justified by the principle of reflection (see Basic Logic)

$\Gamma, (\exists x \in D)\, A(x) \vdash \Delta$ iff for every $d \in D$, $\Gamma, A(d) \vdash \Delta$

iff $\Gamma, z \in D, A(z) \vdash \Delta$

Γ, Δ independent of z

free parameter on propositions, including $\exists$

2. set theory as in Martin-Löf

the explanation of Σ requires a free parameter on sets, including Σ <u>and</u> families depending on Σ

$$\frac{A \text{ prop}}{A \text{ set}}$$

Advantages

- choice sequences

- communication

$\exists$- formation

$$\frac{\Gamma,\ z \in D,\ A(z) \vdash \Delta}{\Gamma,\ (\exists x \in D)\, A(x) \vdash \Delta}$$

$\exists$- reflection

implicit

$$\frac{\Gamma,\ (\exists x \in D)\, A(x) \vdash \Delta}{\Gamma,\ z \in D,\ A(z) \vdash \Delta}$$

trivialize $\downarrow$ $\qquad$ $\uparrow$ compose

axiom

$$z \in D,\ A(z) \vdash (\exists x \in D)\, A(x)$$

compose $\downarrow$ $\qquad$ $\uparrow$ trivialize

explicit

$$\frac{\Gamma \vdash z \in D \qquad \Gamma' \vdash A(z)}{\Gamma, \Gamma' \vdash (\exists x \in D)\, A(x)}$$